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Stress dependency on ultrasonic wave propagation velocity

Part 1 *Analysis by the Eulerian viewpoint of ultrasonic wave velocity in the uniformly deformed isotropic solid*

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The velocity of an ultrasonic wave propagating in the uniformly deformed isotropic solid was analysed by the Eulerian viewpoint. The pseudo elastic coefficient (*PEC*) was used to solve the equation of motion of the elastic wave under finite deformation. The infinitesimal displacement gradients are connected to the stress increments by the *PEC*. Using the *PEC* and the partial differential equation of motion, the velocity of ultrasonic wave was quantitatively related to applied stress, moreover, the stress dependence on longitudinal and transverse wave velocities propagating in the direction parallel or perpendicular to the uniaxial tensile direction could be cleared. Consequently, the Murnaghan's third order elastic constants can be calculated by precisely measuring the uniaxial tensile stress and ultrasonic wave velocity.

Nomenclature

a_1, a_2, a_3	Coordinate of the non-deformed state
X_1, X_2, X_3	Coordinate of the statically and finitely deformed state
x_1, x_2, x_3	Coordinate of the state where dynamic and infinitesimal deformations are superposed on the finitely deformed state
U_1, U_2, U_3	Finite displacement
u_1, u_2, u_3	Infinitesimal displacement
$\partial U_i/\partial X_j$	Finite displacement gradient
$\partial u_i/\partial x_j$, $\{\partial u/\partial x\}_i$	Infinitesimal displacement gradient and its abbreviated formulation
$\mathring{\varepsilon}_{ij}$, ε_{ij}	Eulerian strain in the statically and finitely deformed state and the strain in the state where dynamic infinitesimal deformations are superposed
$\Delta\varepsilon_{ij}$	Strain increment
ε'_{ij}	$1/2(\partial ui/\partial x_j + \partial u_j/\partial x_i)$
S_{ij}	$1/2(\partial U_i/\partial X_j + \partial U_j/\partial X_i)$
I_1, I_2, I_3	Strain invariants
ϱ_0, ϱ, $\Delta\varrho$	Density in the non-deformed state and deformed state and density increment respectively
$\varrho_0\phi$	Free energy per unit volume
λ, μ	Lamé constants
l, m, n	Murnaghan's third order elastic constants
K	$= -\lambda + 3l + m$
L	$= 2\lambda + 2\mu + m$
δ_{ij}	Kronecker's delta
Δ_{ij}	Cofactor of matrix $[\varepsilon_{ij}]$
$\mathring{I}_1$, $\mathring{I}_2$ and $\mathring{\Delta}_{ij}$	Strain invariants and cofactor in the finitely deformed state
I_s	$= S_{11} + S_{22} + S_{33}$
$\mathring{\sigma}_{ij}$, σ_{ij}	Stress in the finitely deformed state and the stress in the state where dynamic and infinitesimal deformations are superposed
$\Delta\sigma_{ij}$, $\{\Delta\sigma\}_i$	Stress increments and their abbreviated formulation
M_{ijkl}, M_{ij}	Pseudo elastic coefficients and their abbreviated formulation
$\mathring{J}$	Jacobian matrix in the finitely deformed state
ω, k	Angular frequency and wave number
ε_1, ε_t, ε	Longitudinal and transverse strain to the tensile axis and strain by hydrostatic pressure
V_{ij}	Ultrasonic wave velocity
E	Young's modulus
v	Poisson's ratio
$\alpha, \beta, \ldots$	Greek subscript letters indicate the summation over all of 1, 2 and 3, for example $\varepsilon_{i\alpha}\varepsilon_{\alpha j} = \varepsilon_{i1}\varepsilon_{1j} + \varepsilon_{i2}\varepsilon_{2j} + \varepsilon_{i3}\varepsilon_{3j}$

1. Introduction

The relationship between stress and strain in the infinitesimal deformed solid can be described by second order elastic (SOE) constants. However it is nonlinear in the finite deformation and needs higher order elastic constants in addition to the SOE ones. The higher order elastic constants were founded by Murnaghan [1]. He applied the mass conservation's law and the principle of virtual work to the law of thermodynamical energy conservation on the isotropic solids, and derived the relation between applied stress and free energy in the finite deformed state. This was used to make a connection between the stress and strain, because the free energy consisted of a polynominal expression of the strains with the SOE and TOE (third order elastic) constants. Moreover, Murnaghan estimated the higher order elastic terms from Bridgman's data of high pressure experiments [2, 3].

Meanwhile, Lazarus [4] estimated the higher order terms by using the change of the ultrasonic propagating velocity in materials under applied stress. Among the higher order elastic constants, the TOE constants are most important to study the properties of solids, not only practically but also theoretically. Hearmon [5] derived the TOE constant of cubic crystals from Lazarus's data.

The higher order elastic constants derived by Murnaghan depend upon the experimental data in the isothermal and static conditions, however those given by the ultrasonic wave are usually obtained through the combined processes of both states of isothermal applied stress and adiabatic ultrasonic wave propagation. The difference between the isothermal elastic constants and the adiabatic ones is small [6, 7], so that the problem is not discussed in this paper.

Hearmon [5], and Hughes and Kelly [8] derived theoretically the stress dependence of the ultrasonic propagating velocity using the TOE constants of cubic crystal, and isotropic solids, respectively. Since then, many researchers have developed these problems [9–14]. Recently the acoustic birefraction theory and the acoustoelastic method, which was analogous to optoelastic method, were proposed [15], and theoretical work concerning the stress dependence on surface wave propagating velocity was also carried out [16]. These theories were applied to the measurement of the TOE constants as well as stress analyses [17, 18].

In the elastic theory, the stress and the equation of motion are based on the strain and free energy. There are two ways of thinking about the strains. One is the Eulerian viewpoint where the coordinates after deformation are taken as a reference, and the coordinates before deformation are differentiated using the reference ones. This is called the Eulerian Strain.

The other is the Lagrangian viewpoint, which is the reciprocal to that of the Eulerian strain. It is possible to neglect the difference between them when the deformation is infinitesimal, but impossible when deformation becomes finite. Consequently the elastic constants should also be distinguished between the Eulerian and the Lagrangian formulae because they are coefficients of the strain in the polynominal expression for the free energy. This results in a difference between the values of the Eulerian and Lagrangian TOE constants, as the expression of strain corresponding to a deformation is not the same. The measurement and theory of the TOE constants by Hearmon and other workers were based on the equation of motion in the Lagrangian viewpoint.

In order to get the Eulerian TOE constants, the relationship of the stress dependence on the ultrasonic propagating velocity has to be derived from the equation of motion in the Eulerian formula. But this equation of motion is expressed by total differentiation [19], and the analysis is not easy. We supposed a pseudo elastic coefficient (*PEC*) to represent the elastic wave of infinitesimal displacement propagating in the statically and finitely deformed solid. The infinitesimal displacement gradients can be linearly connected with the stress increments by the *PEC*.

As a result of using the *PEC*, the equation obtained is not expressed by the total differential term but is only the partial differential formula, even if the Eulerian formula approach was used for the analysis. Using the above process, the secular equation is conducted and stress dependence on the elastic wave propagating velocity could be derived.

2. Aspects of theory

We used the coordinate systems corresponding to each of the three states to treat the elastic wave of infinitesimal deformation propagating in the finitely deformed solid [20]. The first coordinate system corresponds to the undeformed state, the second to the statically, finitely deformed state, and the third to the state when dynamical infinitesimal deformation is superposed on the finite deformation of the second state.

The equation of motion in the Eulerian formula becomes very simple, because the stress tensors are the symmetric Causy tensors, but it is difficult to derive the secular equation from the differential time, since those terms become the total differential time terms in the total differential formula. Therefore we tried to derive a solution from the equation of motion of infinitesimal deformation based on the static, finite deformation state. The total differential terms in the equation of motion of infinitesimal deformation can be replaced by partial differential formulae [21] and this makes the analysis easy. We attempted to linearly connect the stress increments with the strain increments by the infinitesimal deformation, however, only the strain increments, not the stress increments could be expressed in a linear fashion. We then adopted the *PEC* and the infinitesimal displacement gradients instead of the strain increments.

Both the infinitesimal displacement gradients and the stress increments have nine components, accordingly the *PEC* has 81 components, but if the stress increments are symmetrical they become six components in the abbreviated style, then the *PEC* has 54 components.

The differential time term in the equation of motion for the ultrasonic wave superposed on the finite deformation state can be approximately written in the partial differential equation. Using the *PEC* and the partial differential equation, the Eulerian analysis became easy and simple.

3. Pseudo elastic coefficients

3.1. Strain increment and infinitesimal displacement gradient

The coordinates and displacements corresponding to the three states of deformation mentioned above are described as follows using rectangular Cartesian coordinates.

(a) In the case of non-deformed state; coordinate (a_1, a_2, a_3)

(b) In the case of statically and finitely deformed state; coordinate (X_1, X_2, X_3), finite displacement $(U_1, U_2, U_3) = (X_1 - a_1, X_2 - a_2, X_3 - a_3)$

(c) In the case of the state where dynamic and infinitesimal deformations are superposed on the finitely deformed state; coordinate (x_1, x_2, x_3), infinitesimal displacement $(u_1, u_2, u_3) = (x_1 - X_1, x_2 - X_2, x_3 - X_3)$

The strain $\mathring{\varepsilon}_{ij}$ in the statically and finitely deformed state and the strain ε_{ij} in the state where dynamic and infinitesimal deformations are superposed are given as follows by the Eulerian formula,

$$\mathring{\varepsilon}_{ij} = \frac{1}{2}\left(\frac{\partial U_i}{\partial X_j} + \frac{\partial U_j}{\partial X_i} - \frac{\partial U_\alpha}{\partial X_i}\frac{\partial U_\alpha}{\partial X_j}\right)$$

$$\varepsilon_{ij} = \frac{1}{2}\left\{\frac{\partial(U_i + u_i)}{\partial x_j} + \frac{\partial(U_j + u_j)}{\partial x_i} - \frac{\partial(U_\alpha + u_\alpha)}{\partial x_i}\frac{\partial(U_\alpha + u_\alpha)}{\partial x_j}\right\} \quad (1)$$

where subscripts i, j, α and β take either of 1, 2 and 3. We assume that subscript Greek letters indicate summation, but that of Roman letters does not. The state of finite deformation is expressed by sign $^\circ$ above the letter. These descriptions are consistently used in this paper. The strain increment $\Delta\varepsilon_{ij}$ produced at the finitely deformed state is obtained from Equation 1, and written as Equation 2. The derivation of this equation stands on the following assumptions,

(a) terms equal or higher than second order of the infinitesimal displacement gradient $\partial u_i/\partial x_j$ can be neglected,

(b) terms equal or higher than third order products of $\partial u_i/\partial x_j$ and the finite displacement gradient $\partial U_i/\partial X_j$ can be neglected, since $\partial U_i/\partial X_j$ is very small in the elastic limit.

$$\Delta\varepsilon_{ij} = \varepsilon_{ij} - \mathring{\varepsilon}_{ij} = \varepsilon'_{ij} - S_{i\alpha}\frac{\partial u_\alpha}{\partial x_j} - S_{j\alpha}\frac{\partial u_\alpha}{\partial x_i} \quad (2)$$

ε'_{ij} and S_{ij} are defined as a matter of convenience of calculation, and either of them is symmetrical, hence written as

$$\varepsilon'_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

and

$$S_{ij} = \frac{1}{2}\left(\frac{\partial U_i}{\partial X_j} + \frac{\partial U_j}{\partial X_i}\right)$$

S_{ij} is the first order term of the strain at the finitely deformed state. ε'_{ij} corresponds to the strain produced by an infinitesimal deformation superposed on the finitely deformed state. $\Delta\varepsilon_{11}$ and $\Delta\varepsilon_{23}$ are shown as typical examples;

$$\Delta\varepsilon_{11} = \frac{\partial u_1}{\partial x_1} - 2\left(S_{11}\frac{\partial u_1}{\partial x_1} + S_{12}\frac{\partial u_2}{\partial x_1} + S_{13}\frac{\partial u_3}{\partial x_1}\right)$$

$$\Delta\varepsilon_{23} = \frac{1}{2}\left(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}\right) - S_{23}\left(\frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}\right) - \left(S_{22}\frac{\partial u_2}{\partial x_3} + S_{33}\frac{\partial u_3}{\partial x_2} + S_{12}\frac{\partial u_1}{\partial x_3} + S_{13}\frac{\partial u_1}{\partial x_2}\right) \quad (3)$$

$\Delta\varepsilon_{22}$, $\Delta\varepsilon_{33}$, $\Delta\varepsilon_{31}$ and $\Delta\varepsilon_{12}$ can be given by replacing the subscripts of $\Delta\varepsilon_{11}$, $\Delta\varepsilon_{22}$, $\Delta\varepsilon_{23}$ and $\Delta\varepsilon_{31}$ with 2, 3 and 1 instead of 1, 2, and 3, respectively.

3.2. Stress and strain relation

The free energy $\varrho_0\phi$ per unit volume of the deformed isotropic elastic solids is expressed by Murnaghan [1] as follows,

$$\varrho_0\phi = AI_1 + \frac{\lambda + 2\mu}{2}I_1^2 - 2\mu I_2 + lI_1^3 + mI_1I_2 + nI_3 \quad (4)$$

where ϱ_0 is the density in non-deformed state. I_1, I_2 and I_3 are the strain invariants, written as follows;

$$I_1 = \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}$$

$$I_2 = \begin{vmatrix}\varepsilon_{22} & \varepsilon_{23}\\ \varepsilon_{32} & \varepsilon_{33}\end{vmatrix} + \begin{vmatrix}\varepsilon_{33} & \varepsilon_{31}\\ \varepsilon_{13} & \varepsilon_{11}\end{vmatrix} + \begin{vmatrix}\varepsilon_{11} & \varepsilon_{12}\\ \varepsilon_{21} & \varepsilon_{22}\end{vmatrix}$$

$$I_3 = \begin{vmatrix}\varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13}\\ \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23}\\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33}\end{vmatrix}$$

Coefficient A in Equation 4 becomes zero at equilibrium because there is no strain and no applied force, zero strain indicates a local minimum of ϕ.

In this equation, λ and μ are Lamé constants (SOE constants), l, m and n are the TOE constants defined by Murnaghan. Stress is also derived by Murnaghan as follows,

$$\sigma_{ij} = \varrho\left(\frac{\partial\phi}{\partial\varepsilon_{ji}} - 2\varepsilon_{i\alpha}\frac{\partial\phi}{\partial\varepsilon_{j\alpha}}\right) \quad (5)$$

where ϱ is the density of the deformed state. $\partial\phi/\partial\varepsilon_{ij}$ can be obtained from Equation 4, and σ_{ij} is expressed as follows, by neglecting the third order term of strain,

$$\sigma_{ij} = (\lambda I_1 + KI_1^2 + mI_2)\delta_{ij} + (2\mu - LI_1)\varepsilon_{ij} - 4\mu\varepsilon_{i\alpha}\varepsilon_{\alpha j} + n\Delta_{ij} \quad (6)$$

$$K = -\lambda + 3l + m, \quad L = 2\lambda + 2\mu + m$$

where, δ_{ij} is Kronecker's delta and Δ_{ij} is the (ij) cofactor of matrix $[\varepsilon_{ij}]$.

The details of this process are described in Appendix A.

3.3. Relationship between stress increments and infinitesimal displacement gradients

Stress increment $\Delta\sigma_{ij}$ is derived from Equation 6 as

$$\begin{aligned}\Delta\sigma_{ij} &= \sigma_{ij} - \mathring{\sigma}_{ij} = \{\lambda(I_1 - \mathring{I}_1) + K(I_1^2 - \mathring{I}_1^2) \\ &\quad + m(I_2 - \mathring{I}_2)\}\delta_{ij} + 2\mu(\varepsilon_{ij} - \mathring{\varepsilon}_{ij}) \\ &\quad - L(I_1\varepsilon_{ij} - \mathring{I}_1\mathring{\varepsilon}_{ij}) - 4\mu(\varepsilon_{i\alpha}\varepsilon_{\alpha j} - \mathring{\varepsilon}_{i\alpha}\mathring{\varepsilon}_{\alpha j}) \\ &\quad + n(\Delta_{ij} - \mathring{\Delta}_{ij}) \\ &= \delta_{ij}\Bigg[\{\lambda + (2K + m)I_s\}I_1' \\ &\quad - \left(2\lambda\frac{\partial u_\beta}{\partial x_\alpha} + m\varepsilon'_{\beta\alpha}\right)S_{\alpha\beta}\Bigg] - LS_{ij}I_1' \\ &\quad + (2\mu - LI_s)\varepsilon'_{ij} - 2\mu\left\{S_{i\alpha}\left(\frac{\partial u_\alpha}{\partial x_j} + 2\varepsilon'_{\alpha j}\right)\right. \\ &\quad \left. + S_{j\alpha}\left(\frac{\partial u_\alpha}{\partial x_i} + 2\varepsilon'_{i\alpha}\right)\right\} + n\Delta(\Delta_{ij}) \qquad (7)\\ I_1' &= \varepsilon'_{\alpha\alpha}\end{aligned}$$

where, $\mathring{\sigma}_{ij}$, $\mathring{I}_1$ (or $\mathring{I}_2$) and $\mathring{\Delta}_{ij}$ denote stress, strain invariants and cofactor at finitely deformed state, respectively. Equation 7 can be rewritten by use of the above described assumption (a) and (b), then $\Delta\sigma_{11}$ and $\Delta\sigma_{23}$ are

$$\begin{aligned}\Delta\sigma_{11} &= \{\lambda + 2\mu - 2(2\lambda + \mu - 3l - m)I_s \\ &\quad - 2(2\lambda + 7\mu + m)S_{11}\}\frac{\partial u_1}{\partial x_1} \\ &\quad + \{\lambda - 2(2\lambda - 3l - m)I_s - 2\mu S_{11} \\ &\quad + (2\lambda + m + n)S_{33}\}\frac{\partial u_2}{\partial x_2} \\ &\quad + \{\lambda - 2(2\lambda - 3l - m)I_s - 2\mu S_{11} \\ &\quad + (2\lambda + m + n)S_{22}\}\frac{\partial u_3}{\partial x_3} \\ &\quad - (2\lambda + m + n)S_{23}\left(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}\right) \\ &\quad - (2\lambda + 8\mu + m)S_{31}\frac{\partial u_3}{\partial x_1} \\ &\quad - (2\lambda + 4\mu + m)S_{31}\frac{\partial u_1}{\partial x_3} \\ &\quad - (2\lambda + 4\mu + m)S_{12}\frac{\partial u_1}{\partial x_2} \\ &\quad - (2\lambda + 8\mu + m)S_{12}\frac{\partial u_2}{\partial x_1} \qquad (8)\end{aligned}$$

$$\begin{aligned}\Delta\sigma_{23} &= -(2\lambda + 2\mu + m + n)S_{23}\frac{\partial u_1}{\partial x_1} \\ &\quad - (2\lambda + 8\mu + m)S_{23}\left(\frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}\right) \\ &\quad + \left\{\mu - \left(\lambda + 3\mu + \frac{m}{2}\right)I_s\right. \\ &\quad \left. + \left(2\mu - \frac{n}{2}\right)S_{11} - 2\mu S_{22}\right\}\frac{\partial u_2}{\partial x_3} \\ &\quad + \left\{\mu - \left(\lambda + 3\mu + \frac{m}{2}\right)I_s\right. \\ &\quad \left. + \left(2\mu - \frac{n}{2}\right)S_{11} - 2\mu S_{33}\right\}\frac{\partial u_3}{\partial x_2} \\ &\quad - \left(2\mu - \frac{n}{2}\right)S_{12}\frac{\partial u_3}{\partial x_1} - \left(4\mu - \frac{n}{2}\right)S_{12}\frac{\partial u_1}{\partial x_3} \\ &\quad - \left(4\mu - \frac{n}{2}\right)S_{31}\frac{\partial u_1}{\partial x_2} - \left(2\mu - \frac{n}{2}\right)S_{31}\frac{\partial u_2}{\partial x_1} \\ I_s &= S_{11} + S_{22} + S_{33} \qquad (9)\end{aligned}$$

The stress increments $\Delta\sigma_{22}$, $\Delta\sigma_{33}$, $\Delta\sigma_{31}$ and $\Delta\sigma_{12}$ would be given by rewriting the subscripts of $\Delta\sigma_{11}$, $\Delta\sigma_{22}$, $\Delta\sigma_{23}$ and $\Delta\sigma_{31}$, respectively, in the same way for strain increments. The stress increments $\Delta\sigma_{ij}$ can be reduced to six components because of their symmetry. Appendix B shows the derivation of the Equations 8 and 9.

We may express the stress increments as a linear combination of the infinitesimal displacement gradients, and let M_{ijkl}, pseudo elastic coefficients, be their coefficients, so as to get the following equations,

$$\Delta\sigma_{ij} = M_{ij\alpha\beta}\frac{\partial u_\alpha}{\partial x_\beta} \qquad (10)$$

and this is rewritten by the following abbreviated style,

$$\{\Delta\sigma\}_i = [M_{i\alpha}]\left\{\frac{\partial u}{\partial x}\right\}_\alpha$$

$$i = 1, 2, 3, \ldots, 6 \qquad \alpha = 1, 2, 3, \ldots, 9$$

$$\begin{aligned}&\{\Delta\sigma\}_1 = \Delta\sigma_{11}, \quad \{\Delta\sigma\}_2 = \Delta\sigma_{22}, \quad \{\Delta\sigma\}_3 = \Delta\sigma_{33} \\ &\{\Delta\sigma\}_4 = \Delta\sigma_{23}, \quad \{\Delta\sigma\}_5 = \Delta\sigma_{31}, \quad \{\Delta\sigma\}_6 = \Delta\sigma_{12} \\ &\left\{\frac{\partial u}{\partial x}\right\}_1 = \frac{\partial u_1}{\partial x_1}, \quad \left\{\frac{\partial u}{\partial x}\right\}_2 = \frac{\partial u_2}{\partial x_2}, \quad \left\{\frac{\partial u}{\partial x}\right\}_3 = \frac{\partial u_3}{\partial x_3} \\ &\left\{\frac{\partial u}{\partial x}\right\}_4 = \frac{\partial u_2}{\partial x_3}, \quad \left\{\frac{\partial u}{\partial x}\right\}_5 = \frac{\partial u_3}{\partial x_2}, \quad \left\{\frac{\partial u}{\partial x}\right\}_6 = \frac{\partial u_3}{\partial x_1} \\ &\left\{\frac{\partial u}{\partial x}\right\}_7 = \frac{\partial u_1}{\partial x_3}, \quad \left\{\frac{\partial u}{\partial x}\right\}_8 = \frac{\partial u_1}{\partial x_2}, \quad \left\{\frac{\partial u}{\partial x}\right\}_9 = \frac{\partial u_2}{\partial x_1} \qquad (11)\end{aligned}$$

for example,

$$\begin{aligned}M_{11} &= \lambda + 2\mu - 2(2\lambda + \mu - 3l - m)I_s \\ &\quad - 2(2\lambda + 7\mu + m)S_{11} \\ M_{49} &= -\left(2\mu - \frac{n}{2}\right)S_{31}\end{aligned}$$

4. The relationship between the ultrasonic wave propagating velocity and stress

4.1. The equation of motion in a finitely deformed state

We now consider the situation where the elastic wave of infinitesimal deformation is propagating in the finitely deformed solids. When there is no body force, the equation of motion of Eulerian formula is given as,

$$\varrho\frac{d^2(U_i + u_i)}{dt^2} = \frac{\partial\sigma_{i\alpha}}{\partial x_\alpha} = \frac{\partial\mathring{\sigma}_{i\alpha}}{\partial x_\alpha} + \frac{\partial(\Delta\sigma_{i\alpha})}{\partial x_\alpha} \qquad (12)$$

$$\varrho = \mathring{\varrho} + \Delta\varrho \fallingdotseq \mathring{\varrho}$$

where, $\mathring{\varrho}$ denotes the density of the finitely deformed state, and $\Delta\varrho$ means the density increment from infinitesimal deformation. Since U_i is a static displacement, its differentiation with respect to time can be eliminated in Equation 12. The difference between the total and partial differential forms of u_i can be neglected due to the infinitesimal displacement, then the left hand side of Equation 12 may be written as $\partial^2 u_i/\partial t^2$. If we assume $\mathring{\sigma}_{ij}$ on the right hand side is static and spacially uniform, $\partial\mathring{\sigma}_{ij}/\partial X_j$ can be eliminated and $\partial\mathring{\sigma}_{ij}/\partial x_j$ is also eliminated by the following transformation,

$$\frac{\partial \mathring{\sigma}_{ij}}{\partial x_j} = \frac{\partial \mathring{\sigma}_{ij}}{\partial X_\alpha}\frac{\partial X_\alpha}{\partial x_j}$$

Therefore, only the stress increment is valid on the right hand side of Equation 12 and the equation of motion is,

$$\mathring{\varrho}\,\frac{\partial^2 u_i}{\partial t^2} = \frac{\partial(\Delta\sigma_{i\alpha})}{\partial x_\alpha} = M_{i\alpha\beta\gamma}\frac{\partial^2 u_\beta}{\partial x_\alpha \partial x_\gamma} \tag{13}$$

The density $\mathring{\varrho}$ has the following relationship with the density ϱ_0 in the non-deformed state,

$$\mathring{\varrho} = \varrho_0/\det \mathring{J} \tag{14}$$

$$\mathring{J} = [\partial X_i/\partial a_j] \tag{15}$$

$\mathring{J}$ is the Jacobian matrix in the finitely deformed state. The higher than second order terms of the finite displacement gradient could be neglected in the expression of the stress increments. Accordingly the Jacobian matrix can be also approximated by the following expression in the equation of motion,

$$\det \mathring{J} \fallingdotseq 1 + I_s \tag{16}$$

Finally, from the all above mentioned matters, Equation 12 is rewritten as follows,

$$\mathring{\varrho}\,\frac{\partial u_i}{\partial t^2} = (1 + I_s)\frac{\partial(\Delta\sigma_{i\alpha})}{\partial x_\alpha}$$

$$= (1 + I_s)M_{i\alpha\beta\gamma}\frac{\partial^2 u_\beta}{\partial x_\alpha \partial x_\gamma} \tag{17}$$

4.2. The ultrasonic wave propagating velocity under uniaxial tensile deformation

Now, the ultrasonic wave propagating velocity under the state of uniaxial tensile deformation is derived as shown in Fig. 1.

(a) The longitudinal plane wave propagating in the direction of x-axis. In this case, the dynamic displacement contains only u_1, and it is expressed as

$$u_1 = A' \exp\left[i(\omega t - k x_1)\right] \tag{18}$$

where A' is the amplitude, ω the angular frequency and k the wave number. From Equations 11 to 16 and using the condition of $u_2 = u_3 = 0$, Equation 19 and 20 are given as

$$\varrho_0\frac{\partial^2 u_1}{\partial t^2} = (1 + \varepsilon_l + 2\varepsilon_t)M_{11}\frac{\partial^2 u_1}{\partial x_1^2} \tag{19}$$

therefore

$$\varrho_0 V_{11}^2 = \lambda + 2\mu - \varepsilon_l(7\lambda + 14\mu - 6l)$$
$$- 2\varepsilon_t(3\lambda - 6l - 2m) \tag{20}$$

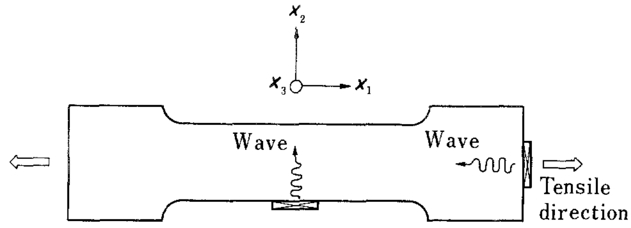

Figure 1 Direction of the ultrasonic wave propagations and tensile deformation. The state of uniaxial tensile deformation; $\mathring{\varepsilon}_{11} = \varepsilon_l$, $\mathring{\varepsilon}_{22} = \mathring{\varepsilon}_{33} = \varepsilon_t$; $\mathring{\varepsilon}_{23} = \mathring{\varepsilon}_{31} = \mathring{\varepsilon}_{12} = 0$; $\mathring{\sigma}_{11}$ = applied stress; $\mathring{\sigma}_{22} = \mathring{\sigma}_{33} = \mathring{\sigma}_{23} = \mathring{\sigma}_{31} = \mathring{\sigma}_{12} = 0$.

where the right hand sides are due to stress increments, and we can assume $\mathring{\varepsilon}_{ij} \fallingdotseq S_{ij}$ because the higher than second order terms of $\partial U_i/\partial X_j$ may be neglected. V_{11} is phase velocity and can be written as $V_{11}^2 = \omega^2/k^2$. The first subscript of V_{11} denotes the direction of ultrasonic wave propagation, and the second one denotes the direction of polarization. If strains ε_l and ε_t almost satisfy the relation $\mathring{\sigma}_{11} = E \cdot \varepsilon_l$ and $\varepsilon_t = -\nu\varepsilon_l$, the ultrasonic wave propagating velocity V_{11}^2 is given as

$$\varrho_0 V_{11}^2 = \lambda + 2\mu - \frac{\mathring{\sigma}_{11}}{E}[7\lambda + 14\mu - 6l$$
$$- 2\nu(3\lambda - 6l - 2m)] \tag{21}$$

where E is Young's modulus

$$E = \frac{\mathring{\sigma}_{11}}{\varepsilon_l} = \frac{\mu(3\lambda + 2\mu)}{(\lambda + \mu)}$$

ν is Poisson's ratio

$$\nu = -\frac{\varepsilon_t}{\varepsilon_l} = \frac{\lambda}{2(\lambda + \mu)}$$

(b) The longitudinal and transverse wave propagating velocity with various combinations of propagation and polarization directions. We obtain the following results,

$$\varrho_0 V_{12}^2 = \mu - \frac{\mathring{\sigma}_{11}}{E}\left[\lambda + 2\mu + \frac{m}{2} - \nu\left(2\lambda + 4\mu + m + \frac{n}{2}\right)\right]$$

$$\varrho_0 V_{21}^2 = \mu - \frac{\mathring{\sigma}_{11}}{E}\left[\lambda + 4\mu + \frac{m}{2} - \nu\left(2\lambda + 2\mu + m + \frac{n}{2}\right)\right]$$

$$\varrho_0 V_{22}^2 = \lambda + 2\mu - \frac{\mathring{\sigma}_{11}}{E}\Big[3\lambda - 6l - 2m - 2\nu(5\lambda + 7\mu - 6l - m)\Big]$$

$$\varrho_0 V_{23}^2 = \mu - \frac{\mathring{\sigma}_{11}}{E}\left[\lambda + \frac{m}{2} + \frac{n}{2} - \nu(2\lambda + 6\mu + m)\right] \tag{22}$$

4.3. The ultrasonic wave propagating velocity in a deformed solid under hydrostatic pressure

In this state the strains and stresses are given as

$$\mathring{\sigma}_{11} \doteqdot (3\lambda + 2\mu)\varepsilon$$

$$\mathring{\sigma}_{11} = \mathring{\sigma}_{22} = \mathring{\sigma}_{33} \neq 0,$$

$$\mathring{\sigma}_{23} = \mathring{\sigma}_{31} = \mathring{\sigma}_{12} = 0$$

$$\varepsilon_{11} = \varepsilon_{22} = \varepsilon_{33} = \varepsilon,$$

$$\varepsilon_{23} = \varepsilon_{31} = \varepsilon_{12} = 0$$

The dynamic displacement of a longitudinal plane wave is

$$u_1 = A' \exp[i(wt - kx_1)] \qquad (23)$$

The propagating velocity V_{11} obtained from *PEC* and M_{11} are related as

$$\begin{aligned} \varrho_0 V_{11}^2 &= (1 + 3\varepsilon)M_{11} \\ &= \lambda + 2\mu - \frac{\mathring{\sigma}_{11}}{3\lambda + 2\mu} \\ &\quad \times (13\lambda + 14\mu - 18l - 4m) \qquad (24) \end{aligned}$$

Similarly, the transverse wave propagating velocity V_{12} is given as

$$\begin{aligned} \varrho_0 V_{12}^2 &= (1 + 3\varepsilon)M_{69} \\ &= \mu - \frac{\mathring{\sigma}_{11}}{3\lambda + 2\mu}(3\lambda + 6\mu + \tfrac{3}{2}m + \tfrac{1}{2}n) \qquad (25) \end{aligned}$$

5. Conclusion

The velocity of an ultrasonic wave propagating in the uniformly deformed isotropic solid was analysed by the Eulerian viewpoint. The coordinate for the analysis was given to each of three states, namely, the undeformed state, the statically finitely deformed state, and the state of dynamic infinitesimal deformation superposed on the finitely deformed state.

The pseudo elastic coefficient was inducted to solve the equation of motion of elastic wave under finite deformation. The infinitesimal displacement gradients are connected to the stress increments by the *PEC*. The *PEC* has 54 components in asymmetry and consists of the second order and third order elastic constants and the strains of finite deformation.

Using the *PEC* and partial differential equation of motion, the velocity of the ultrasonic wave was quantitatively related to applied stress, moreover, the stress dependence on longitudinal and transverse wave velocities propagating in the direction parallel or perpendicular to the uniaxial tensile direction could be cleared. Consequently, Murnaghan's third order elastic constants can be calculated by precisely measuring the uniaxial tensile stress and ultrasonic wave velocity.

Appendix A

Equation 6 can be derived by the following process; using Equation 4

$$\phi = \frac{1}{\varrho_0}\left(\frac{\lambda + 2\mu}{2} I_1^2 - 2\mu I_2 + lI_1^3 + mI_1I_2 + nI_3\right) \qquad (4)$$

where

$$\frac{\partial I_1}{\partial \varepsilon_{ji}} = \delta_{ji}, \; \frac{\partial I_2}{\partial \varepsilon_{ji}} = I_1\delta_{ji} - \varepsilon_{ij}$$

$$\frac{\partial I_3}{\partial \varepsilon_{ji}} = \Delta_{ji}$$

The derivative of Equation 4 by the strain is

$$\begin{aligned} \frac{\partial \phi}{\partial \varepsilon_{ji}} &= \frac{1}{\varrho_0}[\delta_{ji}\{\lambda I_1 + (3l + m)I_1^2 + mI_2\} \\ &\quad + (2\mu - mI_1)\varepsilon_{ij} + n\Delta_{ji}] \qquad (A1) \end{aligned}$$

The ratio of the density at the deformed state to the undeformed one is

$$\frac{\varrho}{\varrho_0} = 1 - I_1 + 2I_2 - 4I_3 \qquad (A2)$$

When Equations are substituted for Equation 5, we obtain Equation 6.

Appendix B

Derivation of the stress increments $\Delta\sigma_{11}$, $\Delta\sigma_{23}$. We obtain the strain increments from Equation 2 as follows,

$$\Delta\varepsilon_{ij} = \varepsilon_{ij} - \mathring{\varepsilon}_{ij} = \varepsilon'_{ij} - S_{i\alpha}\frac{\partial u_\alpha}{\partial x_j} - S_{j\alpha}\frac{\partial u_\alpha}{\partial x_i} \qquad (2)$$

$$\begin{aligned} \Delta I_1 &= I_1 - \mathring{I}_1 = I'_1 - 2S_{\alpha\beta}\frac{\partial u_\beta}{\partial x_\alpha} \\ \Delta(I_1^2) &= I_1^2 - \mathring{I}_1^2 = 2I_sI'_1 \\ \Delta(I_2) &= I_2 - \mathring{I}_2 = I_sI'_1 - S_{\alpha\beta}\varepsilon'_{\beta\alpha} \\ \Delta(I_1\varepsilon_{ij}) &= I_1\varepsilon_{ij} - \mathring{I}_1\mathring{\varepsilon}_{ij} = S_{ij}I'_1 + I_s\varepsilon'_{ij} \\ \Delta(\varepsilon_{i\alpha}\varepsilon_{\alpha j}) &= \varepsilon_{i\alpha}\varepsilon_{\alpha j} - \mathring{\varepsilon}_{i\alpha}\mathring{\varepsilon}_{\alpha j} = S_{i\alpha}\varepsilon'_{\alpha j} + S_{\alpha j}\varepsilon'_{i\alpha} \\ \Delta(\Delta_{11}) &= \Delta_{11} - \mathring{\Delta}_{11} \\ &= S_{22}\varepsilon'_{33} + S_{33}\varepsilon'_{22} - S_{23}\varepsilon'_{32} - S_{32}\varepsilon'_{23} \qquad (B1) \end{aligned}$$

$$\begin{aligned} \Delta(\Delta_{23}) &= \Delta_{23} - \mathring{\Delta}_{23} \\ &= -S_{11}\varepsilon'_{32} - S_{32}\varepsilon'_{11} + S_{12}\varepsilon'_{31} + S_{31}\varepsilon'_{12} \qquad (B2) \end{aligned}$$

$$I'_1 = \varepsilon'_{\alpha\alpha}$$

From the above formulae for the strain increments and Equation 8, the stress increments are

$$\begin{aligned} \Delta\sigma_{ij} &= \delta_{ij}\left[\{\lambda + (2k + m)I_s\}I'_1 \right. \\ &\quad \left. - \left(2\lambda\frac{\partial u_\beta}{\partial x_\alpha} + m\varepsilon'_{\beta\alpha}\right)S_{\alpha\beta}\right] - LS_{ij}I'_1 \\ &\quad + (2\mu - LI_s)\varepsilon'_{ij} - 2\mu\left\{S_{i\alpha}\left(\frac{\partial u_\alpha}{\partial x_j} + 2\varepsilon'_{\alpha j}\right)\right. \\ &\quad \left. + S_{j\alpha}\left(\frac{\partial u_\alpha}{\partial x_i} + 2\varepsilon'_{i\alpha}\right)\right\} + n\Delta(\Delta_{ij}) \qquad (B3) \end{aligned}$$

as in Equation 7.

$\Delta\sigma_{11}$ and $\Delta\sigma_{23}$ are given by rewriting subscripts of ij in the Equation B3 to 11 or 23 and substituting by the Equations B1 and B2.

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